Interpretation of the Normalizable State in the Lee Model with Form Factor

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It is found that in the Lee model with cutoff function, both bound state and resonances can be simultaneously present. The interpretation of the bound state as the physical V particle is discussed. It is made plausible that the bound state in certain cases is a composite particle.

I. INTRODUCTION

THE Lee model has been investigated by many authors with particular regard to its field-theoretic implications on a meaningful definition of mass and charge renormalization.[1,2] Also, the possibility of defining unstable particles has been considered at length in the literature in the framework of such a model.[3-6]

To avoid the appearance of negative probabilities (ghost states), one usually introduces a cutoff function in the interaction. The cutoff function, on the other hand, may also be thought of as representing the nonlocality of the interaction which is due to virtual transitions via couplings not explicitly introduced into the theory.

So far, the consequences of the presence of the cutoff function have not been dealt with in the literature. In particular, the Lee model has been always thought of as able to give either one bound state or one resonance. In the former case, that state has been identified with the stable V particle; in the latter the resonance, if sufficiently narrow, has been considered as an unstable physical V particle.

In this paper, we show that for any reasonable choice of the cutoff function, it is possible to produce simultaneously both bound state and resonances. This is shown in Sec. II by consideration of the structure of the V particle propagator. Having particularized the cutoff function, in Sec. III consideration has been given to the analytic continuation of the propagator to complex energies, for the case in which the mass of the bare V particle is greater than the sum of the masses of the N and θ particles. It is found that it is possible, with a suitable choice of the parameter in the cutoff function, to produce simultaneously both a bound state and a sharp resonance. In Sec. IV we discuss this point with particular regard to the interpretation of the bound state as the physical V particle. It is made plausible that the bound state in certain cases is a composite system. Some mathematical details are presented in three Appendixes.

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II. GENERAL CONSIDERATIONS

The Hamiltonian of the Lee model reads as follows[7]:

$$H=H_0+H_I\,,$$

$$H_0=m_V^{(0)}\int\psi_V^\dagger(\mathbf{p})\psi_V(\mathbf{p})d^3p+m_N\int\psi_N^\dagger(\mathbf{p})\psi_N(\mathbf{p})d^3p+\int\omega(q)a^\dagger(\mathbf{q})a(\mathbf{q})d^3q\,,\tag{2.1}$$

$$H_I=-\frac{g}{(2\pi)^{3/2}}\int\frac{f(\omega)}{(2\omega)^{1/2}}\delta(\mathbf{p}-\mathbf{p}'-\mathbf{q})\times[\psi_V^\dagger(\mathbf{p})\psi_N(\mathbf{p}')a(\mathbf{q})+\psi_N^\dagger(\mathbf{p}')\psi_V(\mathbf{p})a^\dagger(\mathbf{q})]\times d^3p d^3p' d^3q\,.$$

The fermion fields $\psi_V(\mathbf{p})$, $\psi_N(\mathbf{p})$ and the boson field $a(\mathbf{q})$ belong to the particles V, N, and θ, respectively, and satisfy the usual commutation or anticommutation relations. $m_V^{(0)}$, m_N are the bare masses of the V and N particle, respectively; $\omega(q)\equiv\omega=(q^2+\mu^2)^{1/2}$, μ being the mass of the θ particle; g is the real unrenormalized coupling constant, and $f(\omega)$ is the cutoff function. At zero kinetic energy, the cutoff function will be normalized to unity and $\lim_{\omega\to\infty}f(\omega)=0$ will be assumed.

We shall restrict ourselves to the consideration of the invariant manifold consisting of one bare V particle at rest or of the N-θ pair. Owing to the presence of the cutoff function, all physical variables can be expressed in terms of unrenormalized quantities in a well-defined manner, and there is no point in going through the renormalization procedure in the chosen sector.

In the N-θ collisions only the S wave is scattered.[8] The evaluation of the scattering matrix follows once the propagator $S_V'(E)$ of the V particle is known. We have

$$S_V'(E)\equiv\langle0|\psi_V\frac{1}{E+i\epsilon-H}\psi_V^\dagger|0\rangle=\left[E-m_V^{(0)}-\frac{g^2}{4\pi^2}\int_0^\infty\frac{q'^2f^2(\omega')}{\omega'(E+i\epsilon-m_N-\omega')}dq'\right]^{-1},\tag{2.2}$$

[7] Natural units $\hbar=c=1$ will be used.
[8] The scattering is still isotropic when the recoil of the N particle is taken into account [replace m_N with $(q'^2+m_N^2)^{1/2}$ in (2.3)].

where $|0\rangle$ is the vacuum state and $E=m_N+\omega$; the limit $\epsilon\to 0^+$ is understood. The scattering matrix turns out to be

$$S(E)=\frac{[S_V'^*(E)]^{-1}}{[S_V'(E)]^{-1}}$$
$$\equiv\left(E-m_V^{(0)}-\frac{g^2}{4\pi^2}\int_0^\infty \frac{q'^2f^2(\omega')}{\omega'(E-i\epsilon-m_N-\omega')}dq'\right)\Bigg/\left(E-m_V^{(0)}-\frac{g^2}{4\pi^2}\int_0^\infty \frac{q'^2f^2(\omega')}{\omega'(E+i\epsilon-m_N-\omega')}dq'\right). \quad (2.3)$$

The poles of $S_V'(E)$ for $E<m_N+\mu$ correspond to normalizable solutions in the considered sector of the Hilbert space.[9] A necessary condition to have a bound state embedded in the continuum is that $f(\omega)=0$ at that point. To exclude this possibility, we shall suppose in the following that $f(\omega)>0$ for $\omega\geqslant\mu$.

The resonances are obtained for $\mathrm{Re}[S_V'(E)]^{-1}=0$ since $S(E)=-1$ in that case. We shall now pay particular attention to this equation. For the sake of convenience let us define the function $\Phi(E)$ as

$$\Phi(E)=\mathrm{Re}[S_V'(E)]^{-1}+m_V^{(0)}-E$$
$$=-\frac{g^2}{4\pi^2}\mathcal{P}\int_0^\infty \frac{q'^2f^2(\omega')}{\omega'(E-m_N-\omega')}dq', \quad (2.4)$$

where $\mathcal{P}$ denotes the Cauchy principal value for the integral. The function $\Phi(E)$ coincides with that defined by the principal value of the integral of Eq. (7) in Ref. 3. The intersections of $\Phi(E)$ with the straight line $m_V^{(0)}-E$ give rise to resonances if $E\geqslant m_N+\mu$, and to bound states if $E<m_N+\mu$.

From the existing literature[10] one gets the impression that $\Phi(E)$ is a monotonically increasing function of E. As a consequence, one would believe that there is always either one bound state or one resonance. It is true that for $E<m_N+\mu$, $\Phi(E)$ is a monotonically increasing function of E which starts from the value zero at $E=-\infty$. On the contrary, for $E>m_N+\mu$, close inspection of the integral (2.4) shows that for any reasonable choice of the cutoff function, the function $\Phi(E)$ which is positive at $E=m_N+\mu$ tends to zero from negative values as $E\to+\infty$:

$$\lim_{E\to+\infty}\Phi(E)=0^-. \quad (2.5)$$

This is trivial if the cutoff function vanishes identically for $\omega>\omega_{\max}$. However, (2.5) holds at least in the hypothesis that there exists a $\tilde\omega$ above which the function $A^2(\omega)=(\omega^2-\mu^2)^{1/2}f^2(\omega)$ is monotonically decreasing and is dominated by the function B/ω^2 with a suitable positive constant B. This is shown in Appendix I.

It follows that $\Phi(E)$ must vanish at least once for $E>m_N+\mu$. As a consequence, if the condition for the existence of a bound state

$$\Phi(m_N+\mu)>m_V^{(0)}-m_N-\mu, \quad (2.6)$$

is satisfied, we can have either one bound state and no resonance or *one bound state and two (or more) resonances*. If (2.6) is not satisfied we can still get *more than one resonance*. In both cases, low-energy resonances can appear. These reasonings are substantiated by Fig. 1, where two typical behaviors of $\Phi(E)$ are plotted for a smoothly varying cutoff function, together with the straight line $m_V^{(0)}-E$.

Since $\Phi(E)$ is proportional to g^2 for fixed E, when g^2 is varied $\Phi(E)$ changes only in magnitude and not in shape. In particular, the intersection with the E axis and the positions of its stationary points remain unchanged. We distinguish two situations:

(I) $m_V^{(0)}\leqslant m_N+\mu$.

In this case we always have a bound state, since Eq. (2.6) is satisfied for any value of the coupling constant. As g^2 is increased, the bound state starts from the position of the bare V particle, $E=m_V^{(0)}$, and continuously shifts its position towards negative energies. Moreover, from a certain value of g^2 on we will have resonances.

(II) $m_V^{(0)}>m_N+\mu$.

In this case, for $g^2=0$ we evidently have a bound state embedded in the continuum for $E=m_V^{(0)}$. As g^2 is increased the bound state disappears, and reappears for $E<m_N+\mu$ when g^2 is such that (2.6) is satisfied.[11] For $g^2=0$, the propagator $S_V'(E)$ has a pole at $E=m_V^{(0)}$, and this pole disappears as soon as $g^2\neq 0$. For g^2 such that (2.6) is satisfied, the propagator has a pole again, right at the position of the bound state.

The question naturally arises whether the pole which disappeared at the position $E=m_V^{(0)}$ has migrated onto the second lead of the energy Riemann surface, reappearing at the position of the bound state for $E<m_N+\mu$, or whether the bound state is due to a pole of the propagator of different origin.

More insight into this problem can be obtained by considering the case where the zero of $\Phi(E)$ occurs for $E=\bar E<m_V^{(0)}$. In such a case (see Fig. 2), as soon as g^2 is different from zero, a resonance starts from $E=m_V^{(0)}$ and moves continuously towards $E=+\infty$. We are, therefore, inclined to believe that this resonance is due

[9] It can be easily seen that on the first leaf of the energy Riemann surface the singularities of $S_V'(E)$ lie only on the real axis.

[10] See, for example, S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson, and Company, Evanston, Illinois, 1961), Sec. 12b; see also Ref. 3.

[11] We note again that $f(\omega)$ does not vanish for $\omega\geqslant\mu$, so we cannot produce a bound state embedded in the continuum for $g^2\neq 0$.

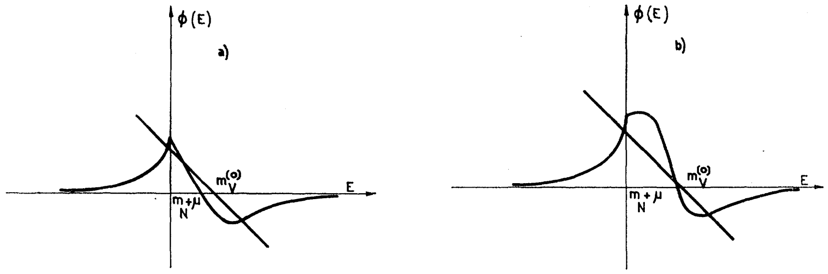

FIG. 1. Two typical behaviors of $\Phi(E)$ as a function of E. The straight line represents the function $m_V^{(0)}-E$. The intersections of $\Phi(E)$ with this line correspond to resonances if they occur for $E\geqslant m_N+\mu$, to bound states if they occur for $E<m_N+\mu$.

to the pole of the propagator which was lying at the position of the bare V particle, $E=m_V^{(0)}$, for $g^2=0$. Consequently, at least in this case, we have good reason to think that the pole corresponding to the bare V particle cannot be associated with the bound state which is produced for $E<m_N+\mu$ when g^2 is such that (2.6) is satisfied. This situation is quite general, largely independent of the choice of the cutoff function.

In the next section we shall investigate this problem in greater detail by making a simple choice for the cutoff function. Moreover, since the use of relativistic kinetic energy for the θ particle introduces complications which are extraneous to the situation discussed above, we will assume, for simplicity, nonrelativistic kinematics.

III. QUANTITATIVE INVESTIGATION OF A PARTICULAR CASE

The transition to nonrelativistic field theory is obtained by simply replacing $(2\omega)^{1/2}$ with $(2\mu)^{1/2}$ in the equation for H_I and placing $\omega(q)=(q^2/2\mu)+\mu$ in H_0. Assuming for the cutoff function the following, physically acceptable, expression:

$$f(\omega)=M^2/(M^2+q^2)\,, \tag{3.1}$$

we obtain for $\Phi(E)$:

$$\Phi(E)=\frac{g^2M^3}{8\pi}\frac{M^2-q^2}{(M^2+q^2)^2} \quad \text{if} \quad E>m_N+\mu$$

$$=\frac{g^2M^3}{8\pi}\frac{1}{(M+|q|)^2} \quad \text{if} \quad E<m_N+\mu\,. \tag{3.2}$$

q^2 and E are related through

$$E=(q^2/2\mu)+m_N+\mu\,.$$

We note that the left derivative of $\phi(E)$ at $E=m_N+\mu$ is $+\infty$, while the right derivative is always finite and negative.

In this section we shall be concerned only with the case $m_V^{(0)}>m_N+\mu$. Defining

$$q_0^2=2\mu(m_V^{(0)}-m_N-\mu)\,, \tag{3.3}$$

we can distinguish three cases:

(i) $q_0^2>M^2$: We have that $\Phi(E)$ vanishes at $E=\bar{E}<m_V^{(0)}$. Note that $\bar{E}$ is a function of M^2. The derivative of $\Phi(E)$ at $E=m_N+\mu$ is less than -1 for the value of g^2 above which we have a bound state, i.e., when the equality sign holds in (2.6), which means

$$(g^2/4\pi)\mu M=q_0^2. \tag{3.4}$$

As g^2 varies we get one resonance in the energy region above $m_V^{(0)}$. This resonance starts at $E=m_V^{(0)}$ for $g^2=0$ and travels towards $E=+\infty$. When (3.4) is satisfied, one bound state and another resonance (at low energy) appear.

(ii) $3q_0^2>M^2>q_0^2$: We now have that $\Phi(E)$ vanishes at $E=\bar{E}>m_V^{(0)}$. The right derivative of $\Phi(E)$ at $E=m_N+\mu$ is again less than -1 for g^2 such that (3.4) is satisfied. As g^2 varies we produce first one resonance, then one bound state and two resonances in the interval $m_N+\mu\leqslant E<m_V^{(0)}$ (note that one resonance is situated at low energy), then one bound state and no resonance, finally one bound state and two resonances in the energy region above $\bar{E}$.

(iii) $M^2>3q_0^2$: $\phi(E)$ vanishes at $E=\bar{E}>m_V^{(0)}$. In this case, the right derivative of $\phi(E)$ at $E=m_N+\mu$ is greater than -1 for that value of g^2 such that (3.4) is satisfied. We have in sequence the following occurrences as g^2 is varied away from zero: one resonance in the interval $m_N+\mu\leqslant E\leqslant m_V^{(0)}$, then one bound state and no resonance, finally one bound state and two resonances which lie at the right of $\bar{E}$.

In Fig. 3 the three cases are sketched. In Appendix II we have taken into consideration the motion of all the stationary points of $|1-S(E)|^2$, also those which are

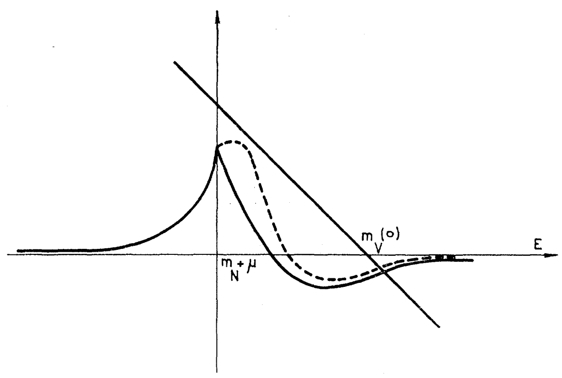

FIG. 2. Plot of $\Phi(E)$; the continuous curve refers to the case (a) of Fig. 1, the dashed curve to the case (b). The straight line $m_V^{(0)}-E$ is also represented. Since $\Phi(E)$ is proportional to the square of the coupling constant, it is seen that when the coupling constant is varied from zero on, there is a resonance which starts from $E=m_V^{(0)}$ and moves continuously to the right.

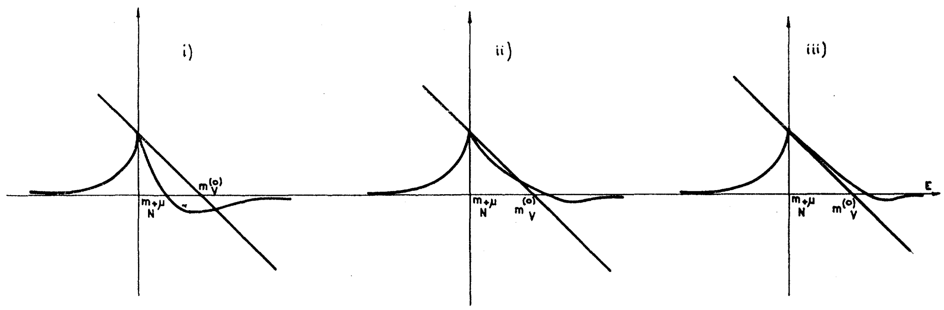

FIG. 3. Plot of $\Phi(E)$ for three different choices of the parameters of the Lee model with cutoff function $f(\omega)=M^2/(M^2+q^2)$. The straight lines represent the function $m_V^{(0)}-E$. The three graphs refer to the following cases: (i) $q_0^2>M^2$. In this case the zero of $\Phi(E)$ occurs for $E<m_V^{(0)}$. (ii) $3q_0^2>M^2>q_0^2$. In this case the zero of $\Phi(E)$ occurs for $E>m_V^{(0)}$ and the right derivative of $\Phi(E)$ at the point $m_N+\mu$ is less than -1. (iii) $M^2>3q_0^2$. In this case the zero of $\Phi(E)$ occurs for $E>m_V^{(0)}$ and the right derivative of $\Phi(E)$ at the point $m_N+\mu$ is greater than -1.

not necessarily resonances (also the case $m_V^{(0)}\leqslant m_N+\mu$ has been considered).

In order to answer the question raised in Sec. II, we shall now investigate the analytic continuation of the propagator in the complex q plane. $S_V'(q)$ (using the same symbol for the propagator when expressed in terms of q) reads as follows:

$$S_V'(q)=-2\mu(q+iM)^2 \times[(q+iM)^2(q_0^2-q^2)+(g^2/4\pi)\mu M^3]^{-1}. \quad (3.5)$$

It can be seen that the poles of $S_V'(q)$ are symmetric with respect to the imaginary axis and that (g being real), in the upper half-plane, they may lie only on the imaginary axis. For $g^2\neq 0$ we have a double zero at $q=-iM$, and four poles. Two of the poles approach, as g^2 vanishes, the points $+q_0$ and $-q_0$, respectively, corresponding to the bare V particle. We shall call them $\mathcal{V}$ poles in the following. The other two, to be referred as $\mathfrak{M}$ poles in what follows, approach $-iM$, where the double zero cancels them. The migration of these poles is studied in detail in Appendix III (also for the case $m_V^{(0)}\leqslant m_N+\mu$), and here we give only the results.

(A) As expected, in the case $q_0^2>M^2$ the $\mathcal{V}$ pole, which for $g^2=0$ is at $q=+q_0$, migrates into the strip $0>\text{Im}q>-\frac{1}{2}M$ of the lower half of the q plane always moving away from the straight line $\text{Re}q=q_0$ and pursuing a path which approaches, for $g^2\to\infty$, the point $q=+\infty-\frac{1}{2}iM$. The $\mathcal{V}$ pole, which for $g^2=0$ is at $q=-q_0$, follows a path symmetric with respect to the imaginary axis. As a consequence, these poles have nothing to do with the bound state which exists when g^2 is such that (2.6) is satisfied. On the contrary, the $\mathfrak{M}$ poles stick to the imaginary axis, one moving upwards giving rise to the bound state as soon as (2.6) is satisfied, the other one falling down to $-i\infty$.

(B) In the case $8q_0^2>M^2>q_0^2$ the situation is analogous to the case $q_0^2>M^2$. The only difference is that the $\mathcal{V}$ poles, instead of moving always away from the imaginary axis, first approach it, but they turn back before reaching it and go, always moving in the strip $0>\text{Im}q>-\frac{1}{2}M$, towards the points $\pm\infty-\frac{1}{2}iM$ as g^2 goes to infinity.

(C) The investigation of Appendix III shows that only when $M^2>8q_0^2$ are the kinematics of the poles radically changed. In fact, in this case the $\mathcal{V}$ poles moving in the strip $0>\text{Im}q>\frac{1}{4}[-M+(M^2-8q_0^2)^{1/2}]$ of the lower half of the complex q plane meet on the imaginary axis at the point $q=\frac{1}{4}i[-M+(M^2-8q_0^2)^{1/2}]$. They immediately depart from one another, one going upwards to produce the bound state, the other going downwards to meet one $\mathfrak{M}$ pole at $q=\frac{1}{4}i[-M-(M^2-8q_0^2)^{1/2}]$. Then these two poles leave the imaginary axis moving in opposite directions in the strip $\frac{1}{4}[-M-(M^2-8q_0^2)^{1/2}]>\text{Im}q>-\frac{1}{2}M$ and approaching, for $g^2\to\infty$, the points $+\infty-iM/2$ and $-\infty-iM/2$, respectively. The other $\mathfrak{M}$ pole moves down to $-i\infty$. The migrations of the poles in the cut E plane are illustrated in Fig. 4.

Since the S matrix is very simply related to the propagator S_V', it is easy to investigate its analytic properties. We have

$$S(q)=\frac{[S_V'(-q)]^{-1}}{[S_V'(q)]^{-1}} \equiv\frac{[(q-iM)^2(q_0^2-q^2)+(g^2/4\pi)\mu M^3](q+iM)^2}{[(q+iM)^2(q_0^2-q^2)+(g^2/4\pi)\mu M^3](q-iM)^2}. \quad (3.6)$$

$S_V'(q)$ satisfies the relation $S_V'^*(-q^*)=S_V'(q)$; the generalized unitarity condition follows:

$$S^*(q^*)S(q)=1. \quad (3.7)$$

Note that $[S_V'(q)]^{-1}$ plays a role quite analogous to the Jost function in potential scattering. From (3.6) we see that the poles of the propagator $S_V'(q)$ are also poles of $S(q)$ and that $S(q)$ has also a "redundant" double pole at $q=iM$. Use of (3.6) shows that the zeros of $S(q)$ appear in positions symmetric to the poles with respect to the real axis. Along any direction in the complex q plane we also have

$$\lim_{q\to\infty} S(q)=1. \quad (3.8)$$

For small q, $S(q)$ behaves like

$$S(q)\underset{q\to 0}{\sim} 1-2q\left[\frac{2\pi i}{\mu g^2}\left(q_0^2-\frac{g^2}{4\pi}\mu M\right)+q+O(q^2)\right]^{-1}. \quad (3.9)$$

From (3.9) we see that $S(q)$ tends to 1 linearly in q ($[S_V'(0)]^{-1}$ finite, zero-energy cross section finite $\neq 0$) unless (3.4) is satisfied, in which case $S(q)$ goes to -1 ($[S_V'(0)]^{-1}=0$) and we have a zero-energy resonance.

It is interesting to make a comparison at this point with the situation which occurs in a separable potential model with the same cutoff function.

The Hamiltonian for the separable potential model is

$$H=\int[m_N+\omega(q)]b^\dagger(\mathbf{q})b(\mathbf{q})d^3q+H_I,$$
$$H_I=-\frac{\lambda^2}{(2\pi)^3\mu^2}\int f(\omega)f(\omega')b^\dagger(\mathbf{q})b(\mathbf{q}')d^3qd^3q', \quad (3.10)$$

where $b(\mathbf{q})$ is the annihilation operator for the $N-\theta$ pair of relative momentum $\mathbf{q}$.

The S matrix for $N-\theta$ collisions reads as follows:

$$S(q)=\frac{[(q-iM)^2+\lambda^2M^3/4\pi\mu](q+iM)^2}{[(q+iM)^2+\lambda^2M^3/4\pi\mu](q-iM)^2}. \quad (3.11)$$

We have the same "redundant" double pole at $q=iM$ as in the Lee model. Besides, we have two poles which move as λ^2 varies and lie always on the imaginary axis at the points

$$q=i[-M\pm\lambda^2M^3/4\pi\mu]. \quad (3.12)$$

For $\lambda^2=0$ these poles are both in $-iM$, and as λ^2 increases one of them moves upwards and produces a bound state when $\lambda^2>4\pi\mu/M$, while the other moves down to $-i\infty$. Note that $S(q)$ satisfies (3.7) and (3.8); for small q it goes to 1 linearly unless $\lambda^2=4\pi\mu/M$, in which case it goes to -1 and we have a zero-energy resonance. A zero-energy bound state is not allowed.

The movement of the poles is strikingly similar to that of the $\mathfrak{M}$ poles of the Lee model in the case $M^2<8q_0^2$. In both models, as the coupling constant is switched off, the pole associated with the bound state, after having travelled on the energy real axis, migrates onto the second leaf of the energy Riemann surface ending at the point $E=-(M^2/2\mu)+m_N+\mu$. The similarity is remarkable if one remembers that in the Lee model there is a V field propagating the $N-\theta$ interaction, while in the separable potential model only the N and θ fields are present, and, therefore, the bound state is a composite particle.

Coming back to the Lee model let us consider the wave function of the bound state of energy $E_b=q_b^2/2\mu+m_N+\mu$ with q_b purely imaginary:

$$|E_b\rangle=\left[\psi_V^\dagger|0\rangle+\frac{g}{[2\mu(2\pi)^3]^{1/2}}\times\int\frac{f(\omega')\psi_N^\dagger a^\dagger(q')}{m_N+\omega'-E_b}|0\rangle d^3q'\right]. \quad (3.13)$$

If $m_V^{(0)}\leqslant m_N+\mu$, as g^2 is switched off adiabatically the wave function of the bound state modifies continuously its structure remaining an eigenfunction of the total Hamiltonian and becoming, when $g^2\to 0$, a bare V particle state. There follows the interpretation of the bound state as the physical V particle. The pole of the propagator $S_V'(E)$ at the point $E=E_b$, as g^2 is switched off, moves continuously over a physical energy region and reaches for $g^2=0$, the position $E=m_V^{(0)}$ mass of the bare V particle. If $m_V^{(0)}>m_N+\mu$ as g^2 is switched off adiabatically at the value of g^2, for which $E_b=m_N+\mu$, the state (3.13) is not normalizable and ceases from being an eigenstate of the total Hamiltonian. Therefore, we cannot perform the limit $g^2\to 0$ on the eigenstate (3.13).[12,13] Thus while in the case $m_V^{(0)}\leqslant m_N+\mu$ the correspondence between the bare and the physical particle is quite clear, in the case $m_V^{(0)}>m_N+\mu$ a correspondence of this kind must be based on different considerations. In this latter case the pole of the propagator associated with the bound state after having traveled on the energy real axis migrates onto an unphysical sheet of the energy Riemann surface. As we have seen, this pole goes towards the points $-M^2/2\mu+m_N+\mu$ or $m_V^{(0)}$, respectively, depending upon whether $M^2<8q_0^2$ or $M^2>8q_0^2$, as $g^2\to 0$. In the case $M^2<8q_0^2$ it never regains a physical meaning. In the case $M^2>8q_0^2$ it gains, in the neighborhood of $g^2=0$, the physical meaning of unstable particle, and for $g^2=0$ the meaning of bare V particle. In one would accept a pole particle association also in the unphysical region of the energy Riemann surface, then one would interpret the bound state of the Lee model associated with the $\mathfrak{M}$ and $\mathfrak{V}$ poles as a composite particle and as the V particle, respectively. We note moreover that there are cases in which the interpretation of the bound state as a composite system is quite natural. In fact when $M^2<q_0^2$ we have a resonance which travels with the $\mathfrak{V}$ pole towards $E=+\infty$. It is undoubtedly correct to state that, at least in this case, the object resonance-$\mathfrak{V}$ pole repre-

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[12] The probability of finding the bare V particle in the bound state $|E_b\rangle$ is given by:

$$P_V=\left[1+\frac{g^2\mu M^3}{4\pi q_b(q_b+iM)^3}\right]^{-1}\equiv\left[1+\frac{q_b^2-2\mu(m_V^{(0)}-m_N-\mu)}{q_b(q_b+iM)}\right]^{-1}.$$

As already pointed out if $m_V^{(0)}\leqslant m_N+\mu$, P_V tends to 1 in the limit $g^2\to 0$ and consequently the bound state is interpreted as the physical V particle. If $m_V^{(0)}>m_N+\mu$ as g^2 is switched off adiabatically, P_V vanishes at $E_b=m_N+\mu$. If P_V is continued analytically to follow the pole of the propagator in the second sheet it immediately obtains unphysical values. It is therefore impossible to stick physical probabilities to the pole associated with the bound state as soon as it migrates onto the second leaf of the energy Riemann surface. Note however that P_V at $g^2=0$ regains a physical value, which is 1 if the bound state is generated by a $\mathfrak{V}$ pole and is zero if the bound state is generated by an $\mathfrak{M}$ pole.

[13] It is interesting to compare the situation now described with that which arises when a finite normalization volume is assumed. As can easily be seen from Fig. 1 of Ref. 3 if $m_V^{(0)}\leqslant m_N+\mu$, in the limit for $g^2\to 0$, E_b, energy of the lowest eigenstate of H, tends to $m_V^{(0)}$ while each "continuum" eigenvalue tends to the eigenvalue of H_0 which lies at its left. On the contrary if $m_V^{(0)}>m_N+\mu$, E_b tends to the head of the "continuum" spectrum of H_0. If we call E_V the eigenvalue of H which lies between the two contiguous eigenvalues of H_0 which enclose $m_V^{(0)}$, when $g^2\to 0$ it is E_V that tends to $m_V^{(0)}$.

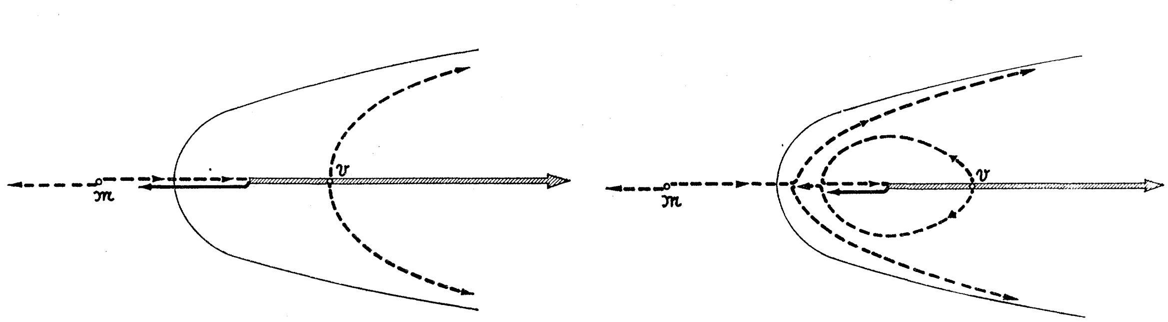

FIG. 4. Migrations of the poles of the propagator, when $m_V^{(0)}>m_N+\mu$, in the cases (A) $q_0^2>M^2$ and (C) $M^2>8q_0^2$ in the complex E-plane cut from $E=m_N+\mu$ to $+\infty$. The positions at $g^2=0$ of the $\mathfrak{M}$ and $\mathcal{V}$ poles are indicated. The dashed paths are in the second leaf of the Riemann surface, the continuous ones are in the first. The lightface curve is the parabola which corresponds to the straight line $q=-\frac{1}{2}iM$ in the q plane and which sets the limits of the allowed region for the poles of the propagator off the real axis on the second leaf of the Riemann surface.

sents an unstable physical V particle. It is true that as g^2 grows bigger the resonance gets larger in width; however, provided that M^2 is suitably chosen, we can still have a sharp resonance for the value of g^2 for which the bound state is produced. The position and width of this sharp resonance will define, as usual, the mass and lifetime of the unstable physical V particle. Therefore, at least in this case, we can draw the conclusion that the bound state of the Lee model is a composite system.[14]

IV. CONCLUSIONS

In Sec. II we have reached quite generally the result that in the Lee model with form factor both bound state and resonances can be simultaneously present. This conclusion raised the problem of the interpretation of the bound state as the physical V particle when $m_V^{(0)}>m_N+\mu$. The quantitative investigations of Sec. III, in particular the conclusions reached for $M^2<q_0^2$, allow us to make some considerations also in the general case when the form factor is not specified, and, therefore, we cannot discuss in detail the path of the poles of the propagator.

By considering the case in which the zero of the function $\phi(E)$ occurs for $E=\bar{E}<m_V^{(0)}$, as already pointed out in Sec. II, we find a resonance which, starting from $E=m_V^{(0)}$ for $g^2=0$, moves towards $E=+\infty$ as g^2 is switched on. We are inclined to think that *the physical V particle is the unstable particle associated with this resonance*. The bound state which appears at the value of g^2 for which (2.6) is satisfied, should therefore be interpreted, at least in this case, as a composite particle. This argument is by all means correct if the resonance is still sharp when the bound state appears. Incidentally, there follows that the distinction between elementary and composite particles cannot be based on the Levinson's theorem, as suggested recently.[15]

[14] Of course, only the shape of the form factor plays the role in the interpretation of the physical structure of the bound state. The coupling constant in fact does not enter there at all.

[15] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).

Going back to our illustrative example we see in fact that to the case $\bar{E}<m_V^{(0)}$ there corresponds the case $M^2<q_0^2$. This is just the condition for the $\mathcal{V}$ pole to move always to the right in the energy plane. The resonance and the $\mathcal{V}$ pole are clearly associated and with them, if M^2 is suitably small, the unstable physical V particle.

Summarizing, we can conclude that there are cases in which the bound state of the Lee model must be considered as a composite system.

APPENDIX I

Let us consider the following integral:

$$\Phi(\omega)=\mathcal{P}\int_\mu^\infty \frac{A^2(\omega')}{\omega'-\omega}d\omega',$$

where $A^2(\mu)=0$. We suppose that there exists a value $\bar{\omega}$ above which: (i) $A^2(\omega)$ is a monotonically decreasing function (we assume also that $dA^2(\omega)/d\omega$ has a limit for $\omega\to\infty$), (ii) $A^2(\omega)\leqslant B/\omega^2$ $(B>0)$. We divide the integration interval in three parts:

$$\Phi(\omega)=\Phi_1(\omega)+\Phi_2(\omega)+\Phi_3(\omega)=\int_\mu^{\bar{\omega}}\frac{A^2(\omega')}{\omega'-\omega}d\omega' +\mathcal{P}\int_{\bar{\omega}}^{2\omega-\bar{\omega}}\frac{A^2(\omega')}{\omega'-\omega}d\omega'+\int_{2\omega-\bar{\omega}}^{\infty}\frac{A^2(\omega')}{\omega'-\omega}d\omega'.$$

We easily see that $\Phi_1(\omega)$ is negative and for $\omega\to\infty$ behaves like $1/\omega$, and that $\Phi_3(\omega)$ is positive but goes to zero like $1/\omega^2$ as ω goes to infinity. Since for $\omega>\bar{\omega}$, $A^2(\omega)$ is a monotonically decreasing function, $\Phi_2(\omega)$ is negative. It follows that $\Phi(\omega)$ is negative in the limit $\omega\to\infty$. By showing that $\Phi_2(\omega)$ vanishes for $\omega\to\infty$, the validity of Eq. (2.5) will be established.

Choosing $a>0$, we have

$$-\Phi_2(\omega)\leqslant\int_{\bar{\omega}}^{\omega-a}\frac{A^2(\omega')}{\omega-\omega'}d\omega'+\mathcal{P}\int_{\omega-a}^{\omega+a}\frac{A^2(\omega')}{\omega-\omega'}d\omega' \leqslant\int_{\bar{\omega}}^{\omega-a}\frac{B}{\omega'^2(\omega-\omega')}d\omega'+\mathcal{P}\int_{\omega-a}^{\omega+a}\frac{A^2(\omega')}{\omega-\omega'}d\omega'.$$

The first term goes to zero as ω goes to infinity. As for the second term, choosing an $\epsilon>0$ arbitrarily small, there exists correspondingly an ω_ϵ such that for $\omega-a>\omega_\epsilon$ the modulus of the derivative $|dA^2(\omega)/d\omega|$ is less than ϵ. It follows that for $\omega>\omega_\epsilon$

$$\mathcal{P}\int_{\omega-a}^{\omega+a}\frac{A^2(\omega')}{\omega-\omega'}d\omega'<\mathcal{P}\int_{\omega-a}^{\omega+a}\frac{A^2(\omega)+\epsilon(\omega-\omega')}{\omega-\omega'}d\omega'=2a\epsilon,$$

as was to be demonstrated.

APPENDIX II

Equation (2.3) for the S matrix reads as follows:

$$S(E)=\frac{E-R(E)-\frac{1}{2}iI(E)}{E-R(E)+\frac{1}{2}iI(E)},\qquad\text{(A1)}$$

where

$$R(E)=m_V{}^{(0)}-\Phi(E)=m_V{}^{(0)}-\frac{g^2M^3}{8\pi}\frac{M^2-q^2}{(M^2+q^2)^2},$$

$$I(E)=\frac{g^2}{2\pi}qf^2(\omega)=\frac{g^2qM^4}{2\pi(M^2+q^2)^2}.$$

Defining $T(E)=1-S(E)$, we get

$$|T(E)|^2=\frac{[I(E)]^2}{[E-R(E)]^2+\frac{1}{4}[I(E)]^2}.$$

We want to study the maxima of $|T(E)|^2$ as a function of E. We have

$$\frac{d|T(E)|^2}{dE}=\frac{2I(E)[E-R(E)]^3}{\{[E-R(E)]^2+\frac{1}{4}[I(E)]^2\}^2}\cdot\frac{d}{dE}\left[\frac{I(E)}{E-R(E)}\right]=0.\qquad\text{(A2)}$$

The zeros of Eq. (A2) occur for $E=R(E)$, in which cases $|T(E)|^2$ assumes the maximum possible value (resonances, see discussion in Sec. III), and for

$$\frac{d}{dE}\left[\frac{I(E)}{E-R(E)}\right]=0.\qquad\text{(A3)}$$

The solutions of this equation are

$$q_{1,2}{}^2=\frac{1}{10}\left\{3q_0{}^2-M^2\pm\left[(3q_0{}^2-M^2)^2+20\left(\frac{g^2\mu M^3}{4\pi}-q_0{}^2M^2\right)\right]^{1/2}\right\},\qquad\text{(A4)}$$

and they produce bumps in $|T(E)|^2$ if they correspond to maxima.

We have the following three possibilities:

(i) If $q_0{}^2>M^2$ we always have one high-energy resonance. For $g^2\mu M/4\pi<q_0{}^2$ we could have, in addition, a bump, given by (A4), which is present anyway for $g^2\mu M/4\pi$ near to $q_0{}^2$. This bump travels to the left and becomes a zero-energy resonance; then a bound state appears together with a resonance which moves, as $g\to+\infty$, towards $\bar{E}$.

(ii) If $3q_0{}^2>M^2>q_0{}^2$, we have first a resonance and possibly a bump, given by (A4), which is anyway certainly present when $g^2\mu M/4\pi\lesssim q_0{}^2$; then the bump becomes a zero-energy resonance, and then a bound-state and a low-energy resonance appear. This resonance merges with the previous one, giving rise to a bump which again increases in magnitude until it becomes a resonance which splits up into two resonances, one travelling towards $\bar{E}$, the other towards $+\infty$.

(iii) If $M^2>3q_0{}^2$, as g^2 varies away from zero, we have in sequence, first a resonance which moves towards $E=m_N+\mu$ and becomes a zero-energy resonance when (3.4) is satisfied, then a bound state and a bump moving to the right and increasing in magnitude until it becomes a resonance which splits up into two resonances, one travelling towards $\bar{E}$ and the other moving towards $+\infty$.

Equation (A4) still holds for the case $m_V{}^{(0)}\leqslant m_N+\mu$, a part the fact that in this case $q_0{}^2\leqslant 0$. Putting $k_0{}^2=-q_0{}^2\geqslant 0$, we see that there is always only one stationary point of $|T(E)|^2$ which is given by:

$$q_2{}^2=\frac{1}{10}\left\{-3k_0{}^2-M^2+\left[(3k_0{}^2+M^2)^2+20\left(\frac{g^2\mu M^3}{4\pi}+k_0{}^2M^2\right)\right]^{1/2}\right\}$$

and travels always to the right for increasing g^2. If we recall the behavior of $\phi(E)$, in this case we conclude that we have first a bump, which travels to the right, increasing until it becomes a resonance. Then the resonance split up in two resonances separated by a minimum. The right resonance and the minimum travel always to the right, the left resonance travels always to the left reaching $E=m_N+\mu$ in the limit $g^2\to+\infty$.

APPENDIX III

Let us consider the equation

$$(q^2-q_0{}^2)(q+iM)^2=g^2\mu M^3/4\pi\qquad\text{(A5)}$$

which determines the poles of the propagator $S_V'(q)$ for the case $m_V{}^{(0)}>m_N+\mu$. Putting $q=x+iy$ we obtain

$$x[(M+2y)x^2-(y+M)(2y^2+My+q_0{}^2)]=0,$$

$$x^4-(6y^2+6My+q_0{}^2+M^2)x^2+(y+M)^2(y^2+q_0{}^2)=g^2\mu M^3/4\pi.$$

In looking for solutions with $x\neq 0$, we get the following equations,

$$x^2=\frac{(y+M)(2y^2+My+q_0^2)}{M+2y}, \tag{A6a}$$

$$\frac{g^2\mu M^3}{4\pi}=-y(y+M)(M+2y)^{-2}[(M+2y)^2+q_0^2]^2. \tag{A6b}$$

Equation (A6b) shows that solutions with $x\neq 0$, $y>0$ are not allowed, i.e., that poles of $S_V'(q)$ can lie in the upper half-plane only on the imaginary axis, as is well known for g real. We now distinguish two cases:

(i) $M^2>8q_0^2$. In this case (A6a) can have solutions only in the strips

$$\tfrac{1}{4}[-M+(M^2-8q_0^2)^{1/2}]<y<0;$$
$$-\tfrac{1}{2}M<y<\tfrac{1}{4}[-M-(M^2-8q_0^2)^{1/2}] \quad \text{or} \quad y<-M;$$

and (A6b) can have solutions only for

$$-M<y<0.$$

Concluding, for $M^2>8q_0^2$ we can have poles with $x\neq 0$ only in the two strips

$$\tfrac{1}{4}[-M+(M^2-8q_0^2)^{1/2}]<y<0,$$

and

$$-\tfrac{1}{2}M<y<\tfrac{1}{4}[-M-(M^2-8q_0^2)^{1/2}].$$

(ii) $M^2<8q_0^2$. (A6a) can have solutions only for

$$y<-M \quad \text{or} \quad -\tfrac{1}{2}M<y<0.$$

The first of these regions is excluded by Eq. (A6b). We can have, therefore, poles with $x\neq 0$ only in the strip

$$-\tfrac{1}{2}M<y<0.$$

Let us now look for poles on the imaginary axis. Equation (A5) reduces to

$$F(y)\equiv(y+M)^2(y^2+q_0^2)=g^2\mu M^3/4\pi. \tag{A7}$$

In plotting $F(y)$ we have two cases:

(i) $M^2>8q_0^2$. $F(y)$ is always positive, with one minimum at $y=-M$ where it vanishes and one maximum and one minimum symmetric with respect to $-\frac{1}{4}M$ at the points

$$y_{1,2}=\tfrac{1}{4}[-M\pm(M^2-8q_0^2)^{1/2}].$$

Note that y_1 is smaller than zero and y_2 is greater than $-\frac{1}{2}M$. The intersections of the straight line $g^2\mu M^3/4\pi$ with $F(y)$ yield the positions of the poles. As g^2 varies away from zero, the poles move in opposite directions away from $-M$ until two new poles appear at the position y_1 when the mentioned straight line is tangent to the minimum. One of these poles goes upwards on the imaginary axis to produce the bound state when g^2 is such that $g^2\mu M^3/4\pi>F(0)$. The other pole goes downwards to meet that coming up from $-M$ and then, when the above-mentioned straight line is tangent to the maximum of $F(y)$, these last two poles disappear from the imaginary axis. Note that when $M^2>[(11+5\sqrt{5})/2]q_0^2$, the bound state is produced before than these poles disappear from the imaginary axis. If $[(11+5\sqrt{5})/2]q_0^2>M^2>8q_0^2$ the afore-mentioned poles have already left the imaginary axis when the bound state is produced.

(ii) $M^2<8q_0^2$. In this case $F(y)$ has only the minimum at $y=-M$ and its shape is parabola-like. For any value of g^2 we have two poles which move away from $-M$. For, $g^2\mu M^3/4\pi>F(0)$ one of them produces a bound state.

It is also interesting to find the intersections, if any, of the poles trajectories with the straight line $y=-x$. If $M^2>(3+\sqrt{8})q_0^2$ we get two intersections; if $M^2<(3+\sqrt{8})q_0^2$ we have no intersection; in this last case, clearly, the pole coming from q_0 turns back before reaching this line. Note that for $M^2=(3+\sqrt{8})q_0^2$ (pole trajectory tangent to $y=-x$) the bound state has not yet been produced by the $\mathfrak{M}$ pole for the value of g^2 such that the $\mathcal{V}$ pole is on $y=-x$.

To study the behavior for large g^2, it is convenient to find the intersections of the pole trajectories with the straight line $y=-\frac{1}{2}M+\epsilon$, with $\epsilon>0$. It is then easy to conclude that as g^2 goes to infinity we have always a couple of poles approaching the points $q=\pm\infty-\frac{1}{2}iM$, respectively.

Consideration of the straight line $x=q_0$ shows that if $M^2>q_0^2$ we have one intersection in the strip $-\frac{1}{2}M<y<0$; if $M^2<q_0^2$ no intersection is obtained.

The movement of the poles in the case $m_V^{(0)}\leqslant m_N+\mu$ is easily obtained from Eqs. (A6a) and (A6b), where $q_0^2=-k_0^2$ has to be assumed negative. We can have poles off the imaginary axis only in the strip

$$-M<y<-M/2.$$

Consideration of Eq. (A7) allows to draw the following conclusions. For $g^2=0$ we have three poles on the imaginary axis, one in $+ik_0$, one in $-ik_0$, and a double one in $-iM$. As g^2 is increased the pole in ik_0 moves always on the imaginary axis towards $+i\infty$. The corresponding pole in $-ik_0$ moves down to $-i\infty$ if $M^2<k_0^2$ in which case the double pole in $-iM$ splits up into two poles which immediately depart from the imaginary axis reaching asymptotically the points $\pm\infty-iM/2$ in the limit $g^2\to+\infty$. If $M^2>k_0^2$ the double pole in $-iM$ splits up into two poles which move on the imaginary axis, one downwards to $-i\infty$, the other upwards to meet the pole moving down from $-ik_0$. Then they immediately depart from the imaginary axis reaching $\pm\infty-iM/2$ in the limit $g^2\to+\infty$.